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LETTER TO THE EDITOR

# Speculation on a scaling law for superconductor–resistor mixture exponent $s$ in a percolation system

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**Abstract.** It is argued that looking for a relationship between the exponent $s$ and the cluster exponents may be more promising than looking for one between the resistor–insulator exponent $t$ and the cluster exponents. We propose $s = \nu - \beta/2$.

It has been a challenge, for more than a decade, to establish a connection between the critical behaviour of cluster properties and of conductivity in percolation systems. (For reviews on percolation problems see Stauffer (1979), Essam (1980) and Deutscher *et al* (1983).) The system under consideration consists of two kinds of conductors, $\sigma_1$ and $\sigma_2$, randomly distributed on the bonds of a $d$-dimensional lattice with probabilities $p$ and $1-p$, respectively. Near to the percolation threshold ($p = p_c$) and to $\sigma_2/\sigma_1 = 0$ the system exhibits critical behaviour:

$$\sigma \propto (p-p_c)^t \qquad \text{for } p>p_c, \qquad \sigma_2 = 0, \qquad 0<\sigma_1<\infty \tag{1a}$$

$$\sigma \propto (p_c-p)^{-s} \qquad \text{for } p<p_c, \qquad \sigma_1 = \infty, \qquad 0<\sigma_2<\infty \tag{1b}$$

$$\sigma \propto (\sigma_2/\sigma_1)^u \qquad \text{for } p=p_c, \qquad (\sigma_2/\sigma_1) \ll 1 \tag{1c}$$

where $\sigma$ is the macroscopic conductivity of the system. The exponents in equations (1) obey the scaling law (Efros and Shklovskii 1976, Straley 1976)

$$u = t/(s+t). \tag{2}$$

The cluster exponents $\alpha$, $\beta$, $\gamma$, $\delta$, $\nu$ defined in the standard way (Stauffer 1979) are also connected by scaling laws

$$2-\alpha = \gamma + 2\beta = \beta(\delta+1) = d\nu. \tag{3}$$

The critical dimensionality $d_c$ for both (conductivity and cluster) problems is 6. The following relationships between the two kinds of exponents have been proposed (Skal and Shklovskii 1974):

$$t = (d-2)\nu + \zeta \tag{4}$$

with

$$\zeta = 1, \qquad 1 \leqslant d \leqslant 6 \tag{5a}$$

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or with (Levinshtein *et al* 1975):

$$\zeta = \nu \qquad \text{if} \qquad d = 2. \tag{5b}$$

A recent hypothesis due to Alexander and Orbach (1982) is

$$t = [\nu(3d-4) - \beta]/2. \tag{6}$$

Formulae (4)–(6) are based on different heuristic pictures on the infinite cluster structure.

Straley (1980) argued that both $t$ and $s$ should be given equally important placement in a 'hyperscaling' law and he suggested

$$d\nu = t + s. \tag{7}$$

In the light of recent numerical work in $d = 2$ (Mitescu *et al* 1982, Derrida and Vannimenus 1982, Li and Strieder 1982a, b) the value of $t(d = 2)$ seems to be below $\nu(d = 2)$ but definitively above 1. Since $u(d = 2) = \frac{1}{2}$ (Dykhne 1970), from (2), $t(d = 2) = s(d = 2)$. Thus for $d = 2$, (4) with (5$b$) and (7) give the result: $t(d = 2) = \nu(d = 2)$.

Equations (4) with (5$a$) lead to $t(d = 2) = 1$, while from (6) one gets $t(d = 2) = 1.264$ ($\nu(d = 2) = \frac{4}{3}$ (den Nijs 1979)). Consequently, at least for $d = 2$, only (6) gives an acceptable result. On the other hand Alexander and Orbach (1982) derived a formula in the same context for the anomalous diffusion on lattice animals which is not in good agreement with recent three-dimensional simulations by Wilke *et al* (1983).

Furthermore, one expects that a general scaling law connecting conductivity and cluster exponents should be valid in $d = 1$ too, since neither (2) nor (3) is violated in $d = 1$. But equation (6) is definitely wrong in $d = 1$; it gives $t = -\frac{1}{2}$, while from (2) $t = 0$ follows, which is the only physical value as no conducting phase exists. (The 'semiderivation' of equation (6) by Wilke *et al* (1983) holds only for $d > 1$.) We conclude that, at the moment, no reliable scaling law relating $t$ to the cluster exponents is known, though equation (6) might finally turn out to be good for $1 < d < 6$. It is doubtful if such a relationship exists at all. The exponent $t$ is defined by the non-analytic behaviour of a transport coefficient (1$a$); thus it is a dynamical critical exponent and as such usually independent of static exponents. At the same time the exponent $s$ describes the divergence of the electric susceptibility $\chi$ too (Efros and Shklovskii 1976):

$$\chi \propto |p - p_c|^{-s}, \tag{8}$$

which is a static quantity. Therefore we hope that looking for a relationship between $s$ and the cluster exponents may be more promising.

The problem of the 'ant in a labyrinth'—a random walk on percolation clusters—is closely related to the conductivity problem (for a review see Mitescu and Roussenq 1983). Below the threshold the ant is always confined within finite clusters and this leads to a finite value of $l^2 = \lim_{\tau\to\infty} \langle (R(\tau) - R(0))^2 \rangle$ where $R(\tau)$ is the position of the ant at time $\tau$ and the brackets stand for configurational average. The quantity $l^2$ diverges for $p \to p_c - 0$

$$l^2 \propto (p_c - p)^{-s'}. \tag{9}$$

Such a diverging 'localisation length' gives rise to a diverging electric susceptibility since $\chi \propto l^2$ (Götze 1978). ($l^2 \propto |p - p_c|^{-s'}$ is valid on both sides of $p_c$ if only finite clusters are taken into account for $p > p_c$.) Thus we have

$$\chi \propto |p - p_c|^{-s'} \tag{10}$$

with

$$s' = 2\nu - \beta \tag{11}$$

from scaling considerations (Stauffer 1979, Vicsek 1982). Identification of $s$ from equation (8) with $s'$ from (11) (Stephen 1978) is, however, misplaced: the divergence in (10) comes from the polarisability of finite clusters, while equation (8) describes the divergence of $\chi$ due to the capacitance *between* the clusters (Gefen *et al* 1983). In $d = 1$ both cases can be followed analytically: $s'(d = 1) = 2$ (Odagaki and Lax 1980) while $s(d = 1) = 1$ as can be easily checked by calculating the effective dielectric constant in a chain containing capacitors with probability $(1-p)$ and resistors with probability $p$ when $p \to 1$. The scaling law, which we want to propose here as a conjecture, is that the relationship

$$s = \tfrac{1}{2}s' = \nu - \tfrac{1}{2}\beta \tag{12}$$

is true for $1 \leqslant d \leqslant 6$. Equation (12) is exact not only at $d = 1$ but also at $d_c = 6$, since here $s = s' = 0$ (Straley 1977). Table 1 compares (12) with available data.

**Table 1.** Dimensionality dependence of $s$, compared with equation (12).

| $d$ | $s$ | $\nu - \beta/2$ |
|---|---|---|
| 1[a] | 1 | 1 |
| 2 | $1.28 \pm 0.02$[b] | $91/72 = 1.264$[c] |
| 3 | $0.70 \pm 0.02$[d]<br>$0.5 \pm 0.1$[e] | $0.66 \pm 0.02$[f] |
| 4 | $0.6 \pm 0.1$[g] | $0.45 \pm 0.1$[h] |
| 6[a] | 0 | 0 |

[a] Exact results.
[b] Binder and Stauffer (1983): $s(d = 2) = t(d = 2)$ and the quoted result is an average over the recent numerical estimates.
[c] den Nijs (1979), Pearson (1980), Nienhuis *et al* (1980).
[d] Simple cubic bond.
[e] Simple cubic site percolation (Straley 1977).
[f] Heermann and Stauffer (1981), Margolina *et al* (1982), Gaunt and Sykes (1983).
[g] Straley (1978).
[h] Kirkpatrick (1976).

For $d = 2$ equation (12) coincides with Alexander and Orbach's (1982) conjecture (6) and is very accurate. For $d = 3$, (12) corresponds well to a weighted average of Straley's (1977) results for bond and site problems. The discrepancies in higher dimensionalities can be due to the increasing numerical errors here. (Note that the uncertainties in table 1 are confidence intervals rather than error limits). Unfortunately we do not have an explanation for equation (12) except for the trivial $d = 1$ case. Further work is needed to decide whether equation (12) is exact or only a good approximation.

After this letter was finished, we learned (Stauffer, private communication) that M Daoud proposed a different relationship between $s$ and the cluster exponents. However, since his formula gives $s = \nu$ in $d = 2$, our hypothesis is clearly superior to that of Daoud, at least in $d = 2$.

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